Optimal digital control of an uncertain parameter 2nd order plant

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Abstract
In the paper a construction of a control system for 2nd order, uncertain-parameter plant is discussed. The considered model of the plant is described by state space equation or by equivalent transfer function and it describes a huge class of real control plants, for example – electric drives or oriented PV systems. As a controller the digital proportional (P) controller was employed. The control system is going to be implemented at the microcontroller platform. Results are by the example depicted.

Key words: Digital control, optimal control, uncertain parameter systems

Introduction

The minimum energy digital control has been one of main problems in control theory since many years. This problem has been presented for example in classic book Athans Falb (1966). Fundamentals of digital control are given by Isermann in 1989 and 1992, the modeling and control of uncertain parameter systems with two-dimensional uncertain parameter space have been analysed by Oprzędkiewicz and they are summarized in monograph from 2009, the control of this class of plants has been analysed for example by Mitkowski and Oprzędkiewicz in 2011 and 2012.

This paper is devoted to present the synthesis of digital control system for a class of uncertain parameter systems described by state equation or equivalent transfer function. The uncertain parameters of the plant are described by interval numbers. The control system is expected to assure the realization of servo control with minimal energy consumption. Results can be employed to control of the oriented PV system or another similar plant.

The paper is organized as follows: at the beginning the considered class of control plants is given. Next the discrete control system and its optimization are discussed. Finally the numerical example and final conclusions are presented.

Materials and Methods

The plant under consideration

The most general model of the plant under consideration is an uncertain-parameter, linear state space equation:

$$\begin{aligned} x(t) &= A(q)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (1)$$

In equation (1) $x(t) \in R^2$, $u(t) \in R$, $y(t) \in R$. A state matrix of the system is a Frobenius matrix, parameter q describes the uncertainty of the model and it is described by the positive interval number.

$$A(q) = \begin{bmatrix} 0 & 1 \\ 0 & -q \end{bmatrix} \quad (2)$$

where:

$$q = [\underline{q}, \overline{q}], \quad \underline{q} > 0 \quad (3)$$

Control and output matrices are well known and they are equal:

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad (4)$$

A transfer function *G(s,q) = C(sI-A(q))-1B* can be assigned with the use of (1) – (4) and it is a function of the uncertain parameter q:

$$G(s,q) = \frac{1}{s^2 + qs} \quad (5)$$

The plant described by (1)-(5) will be controlled with the use of discrete controller. This implies, that A/D and D/A converters working synchronically should be applied. The both elements build so called process interface. The scheme of the plant with both elements is shown in the figure 1.

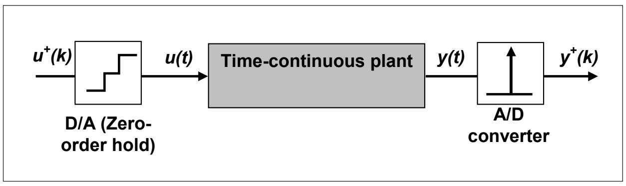

Figure 1. Continuous control plant with D/A and A/D converters

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To describe the discrete plant shown in figure 1 the continuous-time state space equation (1) – (4) should be transformed to the discrete form with the use of following relations:

$$A^+(q) = e^{T_s A(q)}, \quad B^+(q) = \int_0^{T_s} e^{A(q)t} B dt, \quad C^+ = C \tag{6}$$

A discrete transfer function of the plant shown in the Figure 1 is described as follows:

$$G^+(z) = C^+ \left(zI - A^+\right)^{-1} B^+ + D^+ \tag{7}$$

If the plant is described by the continuous transfer function (5), then the discrete transfer function (7) can be expressed as follows:

$$G^+(z) = \left(1 - z^{-1}\right) Z\left(\frac{G(s)}{s}\right) \tag{8}$$

The discrete transfer function (8) is a function of uncertain parameter q and the sample time T_s and it can be written in the form analogical to (5):

$$G^+(z, q, T_s) = \frac{v_1(q, T_s) z + v_0(q, T_s)}{z^2 + w_1(q, T_s) z + w_0(q, T_s)} \tag{9}$$

In (9) v and w denote coefficients of numerator and denominator of (9) respectively. They are equal:

$$\begin{aligned} w_1(q, T_s) &= -\left(1 + e^{-qT_s}\right) \\ w_0(q, T_s) &= e^{-qT_s} \\ v_1(q, T_s) &= \frac{1}{q} + \frac{1}{q^2}\left(1 + e^{-qT_s}\right) - \frac{2}{q^2} \\ v_0(q, T_s) &= \frac{1}{q^2} + \left(\frac{1}{q^2} - \frac{1}{q}\right) e^{-qT_s} \end{aligned} \tag{10}$$

During construction of a discrete system shown in Figure 1 and described by (9) – (10) a crucial problem is the proper selection of the sample time T_s. The properly set sample time should assure the proper work of the control system for each value of uncertain parameter of the plant, described by the parameter q.

The value of sample time T_s is determined both by the plant properties and the applied control algorithm. The sample time assign problem is one of fundamental in Control Theory and it was presented by many Authors, for example by Isermann (1989, 1992). Additionally, in the considered case the uncertainty of the discrete system should not be greater, than the uncertainty of the continuous system (see Oprzędkiewicz 2009).

A closed-loop, discrete P control system

The uncertain-parameter plant described above can be controlled with the use of digital controller. The general scheme of a discrete, closed – loop control system for plant under consideration is shown in the Figure 2.

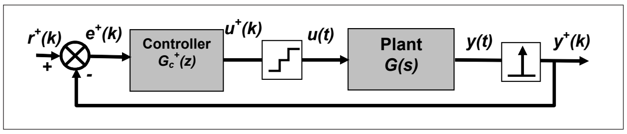

Figure 2. The closed-loop digital control system for the considered plant

In figure 2 $r^+(k)$ denotes a seat point, $e^+(k)$ denotes discrete error, $u^+(k)$ denotes discrete control signal, $u(t)$ denotes continuous control signal, $y(t)$ denotes a continuous process value and $y^+(k)$ denotes the discrete process value.

For the control system shown in figure 2 an analytical relationship between „z" transform of the control system $U^+(z)$, "z" transform of seat point $R^+(z)$ and transfer functions $Gc^+(z)$ and $G^+(z)$ can be given. It has the following form:

$$U^+(z) = \frac{G_c^+(z)}{1 + G^+(z) G_c^+(z)} \cdot R^+(z) \tag{11}$$

Assume, that to the control of the plant an ideal proportional (P) controller, described by the transfer function (12) should be applied:

$$G_c^+(z) = k_r \tag{12}$$

The proportional gain of the controller (12) should be assigned with respect to the following factors:

The closed-loop control system must be stable for each value of uncertain parameter of the system, described by the interval number q.

The cost function (described in the next section) for the control system should be minimal.

With respect to (11) and (12) the discrete control signal can be expressed as follows:

$$U^+(z) = \frac{k_r z^2 + k_r w_1(q, T_s) z + k_r w_0(q, T_s)}{z^2 + \left(w_1(q, T_s) + k_r v_1(q, T_s)\right) z + \left(k_r v_0(q, T_s) + w_0(q, T_s)\right)} \cdot R^+(z) \tag{13}$$

In (13) v and w denote coefficients of numerator and denominator. They are expressed by (10).

Furthermore, the discrete transfer function $G_z^+(z) = Y^+(z)/R^+(z)$ of the discrete control system shown in Figure 2 is a function of uncertain parameter q, sample time T_s and gain k_r:

$$G_z^+(z, q, T_s, k_r) = \frac{Y^+(z)}{R^+(z)} = \frac{k_r v_1(q, T_s) z + k_r v_0(q, T_s)}{z^2 + \left(w_1(q, T_s) + k_r v_1(q, T_s)\right) z + \left(k_r v_0(q, T_s) + w_0(q, T_s)\right)} \tag{14}$$

The characteristic polynomial of the closed-loop system $W(z)$ is also a function of uncertain parameter q, sample time Ts and gain k_r. It is equal:

$$W(z, q, T_s, k_r) = z^2 + \left(w_1(q, T_s) + k_r v_1(q, T_s)\right) z + \left(k_r v_0(q, T_s) + w_0(q, T_s)\right) \tag{15}$$

Roots of polynomial (15) build a spectrum of closed-loop system Λz^+, which can be defined as follows:

$$\Lambda_z^+(q,T_s,k_r)=\{\lambda_i^+(q,T_s,k_r):W(\lambda_i^+,q,T_s,k_r)=0\},\quad i=1,2 \tag{16}$$

The considered closed-loop control system is required to be practically implemented on the microcontroller platform. This implies, that the parameters of the discrete controller: sample time T_s and proportional gain k_r must be possible to physical realization onto particular hardware platform. The most simple notation of these boundaries is to write they as intervals (17):

$$\begin{aligned}&\underline{k_r}\le k_r\le\overline{k_r},\ \underline{k_r}>0\\&\underline{T_s}\le T_s\le\overline{T_s},\ \underline{T_s}>0\end{aligned} \tag{17}$$

The both controller parameters build the vector p defined by (18):

$$p=[k_r\quad T_s]^T \tag{18}$$

where k_r and T_s are parameters possible to technical realization in particular platform. All vectors p build the space of controller parameters P, which is a subspace of R^3:

$$P=\{p=[k_r,T_s]^T:\underline{k_r}<k_r\le\overline{k_r},\underline{T_s}\le T_s\le\overline{T_s}\}\subset R^3 \tag{19}$$

Set P defined by (19) can be interpreted as cube in space R^3.

The spectrum (16) with the respect to (18), (19) can be written as follows:

$$\Lambda_z^+(p,q)=\{\lambda_i^+(p,q):W(\lambda_i^+(p,q))=0,\ p\in P\}\quad i=1,2 \tag{20}$$

Results and Discussion

The stability discussion

The discrete transfer function (14) of the closed-loop control system is a function of parameters collected in the set P and uncertain parameter q. These parameters determine elementary possibilities of the considered control system. It is obvious, that the fundamental property of each control system is their stability.

In the considered case the sufficient and necessary condition of stability is, the whole spectrum of the system must be localized inside the unit circle. The meeting of this condition is determined by values of parameter q and controller parameters $p\in P$ and it can be expected, that for certain values of p and q the control system will be stable and for other it will not be stable.

Vectors p assuring the stability of the system will be denoted by p_s, and set of all vectors p_s will be denoted by P_s. It can be expressed as follows:

$$P_s=\{p\in P:|\lambda_i^+(q,p)|<1\}\quad i=1,2 \tag{21}$$

It is obvious, that $P_s\subseteq P$. The set P_s can be assigned analytically or numerically and it can be interpreted as three dimensional in R^3.

The cost function

For the considered control system the classic servo control problem can be formulated: the process value $y(t)$ should trace the seat point $r(t)$ with minimal possible energy consumption. A function describing the energy consumption in the discrete control system can be most simply expressed as follows:

$$J_1^+(p_s)=\sum_{k=0}^{\infty}(u^+(k))^2 \tag{22}$$

At this moment remember, that the control system is going to be implemented onto microcontroller platform. This assumption can be additionally applied to estimation the energy consumption in the system, because the energy consumption in microcontroller system increases with the speed of work. Furthermore, the unique measure of this speed is the sample time T_s: the smaller sample time denotes the bigger speed of the system.

In conclusion, the cost function for the considered control problem can be proposed as follows:

$$J_2^+(p_s,q)=h_1\sum_{k=0}^{\infty}(u^+(k))^2+h_2\frac{1}{T_s} \tag{23}$$

where:

$u^+(k)$ – the discrete control signal in k – th step,

T_s – the sample time,

h_1, h_2 – weight coefficients.

The cost function (27) is a function of plant parameter q and controller parameters: T_s and k_r.

Weight coefficients in the function (23) are assigned to describe detailed real situation – for example, if the more important is the speed of work the control system, the coefficient h_2 is smaller, than the h_1.

Geometric interpretation of the cost function

The cost function (23) is a function of uncertain plant parameter, described by the interval number q and controller parameters, expressed by vector $p_s \in P_s$. This implies, that the geometric interpretation of this function in the "visible" space R^3 can be not simple. To make the geometric interpretation of function (23) possible to propose we can notice, that:

For the constant, selected value of the uncertain plant parameter q the cost function as a function of set P_s can be interpreted as a surface in the R^3.

Consequently, for all values of uncertain parameter q, described by an interval number $[\underline{q},\overline{q}]$ the cost function can be interpreted as a body in R^3 space, limited by border values of q.

The both above notes allow us to present the geometric interpretation of cost function (23). It is shown (for exemplary values of all parameters) in the figure 3.

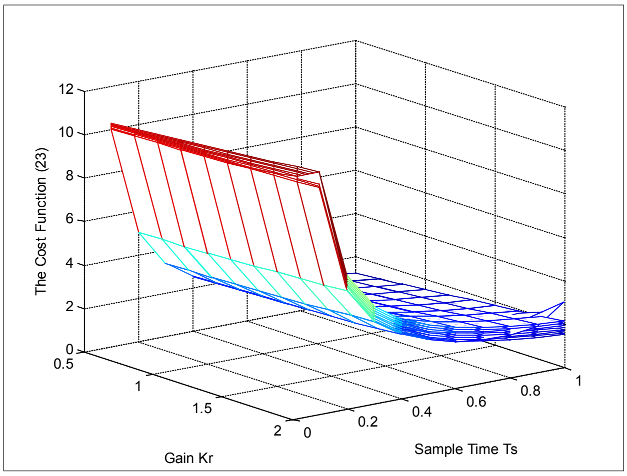

Figure 3. The geometric interpretation of the cost function (23)

Remarks about the synthesis of an optimal controller

After the whole above considerations the discrete control system synthesis problem can be formulated as follows:

Inside the set of controller parameters P_s should be found such values of sample time T_{s0} and gain k_{r0} for which the cost function (23) is minimal for each value of uncertain parameter q:

$$J_2^{+}(p_{s0},q) \le J_2^{+}(p_s,q),\ q \in Q \tag{24}$$

Notice, that for the fixed value of uncertain parameter q the cost function (23) is determined by controller parameters only.

The analytical formula of cost function (23) is possible to formulate, but it is extremally complicated and use it to minimum search is rather impossible. This implies, that the minimization of cost function (23) can be run with the use of numerical methods with support of geometric interpretation of cost function and additionally remark presented underneath.

Remark 1

Assumptions:

Consider the interval dynamic system described by (1) – (5),

The system is controlled with the use of discrete proportional controller with gain k_r and sample time T_s.

The ranges of controller parameters are also described by interval numbers and they generate the set of controller parameters P_s described by (21),

The uncertain parameter of the plant is described by interval number q.

Thesis:

For constant values of sample time T_s and uncertain parameter q the cost function (23) is an increasing function of controller gain k_r

The proof of the above remark is presented underneath.

At the beginning notice, that the cost function (23) can be expressed in the following form:

$$J_2^{+}(p_s,q) = h_1 f_1(p_s,q) + h_2 f_2(p_s) \tag{25}$$

where:

$$f_1(p_s,q) = \sum_{k=0}^{\infty}\left(u^{+}(k)\right)^2 \tag{26}$$

$$f_2(p_s) = \frac{1}{T_s} \tag{27}$$

The function $f_1(p_s,q)$ described by (26) is a function both uncertain parameter q and controller vector p_s, the function $f_2(p_s)$ is a function of sample time T_s only. This implies, that the function of proportional gain k_c is the function (26) only.

Next, in the case of continuous time the function (26) describing the energy consumption is expressed as the following integral:

$$I(q,k_c) = \int_0^{\infty} u^2(t)dt \tag{28}$$

The value of continuous integral (28) can be simply calculated analytically. It is equal:

$$I(q,k_c) = \frac{k_c(k_c+q)}{2q} \tag{29}$$

The function (29) does not have a minimum for positive q and positive k_c and it increases for increasing value of parameter k_c and it decreases for increasing value of the uncertain parameter q.

Furthermore, the continuous function (28) can be interpreted as a boundary case of discrete function (26). It can be expressed as follows:

$$I(q,k_r) = \lim_{T_s \to 0} I^{+}(q,k_r,T_s) \tag{30}$$

Finally, if the continuous function (28) is increasing for positive, increasing k_c and the equation (30) is kept, we can expect, that the discrete function (26) will behave analogically. This conclusion ends the proof.

Example

As an example consider the plant described by transfer function (5) with uncertain parameter described by interval number q = [1.0;2.0]. For this plant we construct the discrete control system shown in figure 2. The set P_s of controller parameters assuring the stability of the closed-loop control system is described as underneath:

$$\begin{aligned} 0.1 \le T_s \le 1.0 \\ 0.5 \le k_r \le 2.0 \end{aligned} \tag{31}$$

At the beginning a distribution of cost function (23) for ranges of controller parameters described by (31) was assigned. The cost function was calculated with the use of (23), both weight coefficients h_1 and h_2 were equal 1.0. The control function was calculated with the use of equation (13), the step amplitude was equal 1.0. The distribution of cost function along the set P_s described by (31) is shown in Figure 3. The vertex values of cost function (23) are written in Table 1.

Table 1. The vertex values of cost function (23) in Example

q	T_s	0.1	**1.0**	1.0	0.1
	k_r	0.5	**0.5**	2.0	2.0
	1.0	10.0355	**1.0490**	3.0280	11.1175
	2.0	10.0602	**1.0639**	1.5317	10.4036

For both values of uncertain parameter q the minimum of the cost function (23) for the example we deal with is localized in vertex $\left[\overline{T}_s;\underline{k}_r\right]$ of the set P_s. The minimal value of for both border values of uncertain parameter q is shown in the Table 1 also.

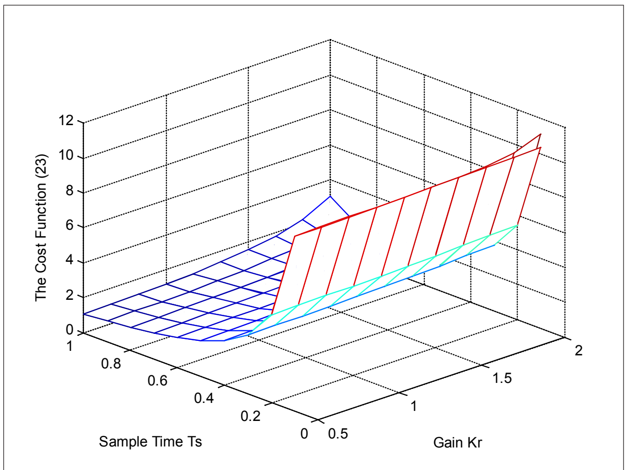

Figure 4. The spatial distribution of the cost function (23) considered in example

From Figure 4 and Table 1 the following conclusions can be formulated:

- The sensitivity of the cost function (23) on the plant uncertainty expressed by the parameter q depends on the controller gain k_c:
  - for smaller values of k_c the sensitivity is small,
  - for bigger values of gain k_c is bigger.
- For values of sample time T_s closer to maximal the cost function (23) has the minimum along the proportional gain k_r.
- For the considered case extremes of cost function are localized only in vertices of considered area of controller parameters P_s.

Conclusions

The final conclusions from the paper can be formulated as follows:

- Results of simulations show, that the sensitivity the considered cost function on uncertainty the control plant, described by parameter q depends on the gain the controller.
- The geometric interpretation of the cost function simplifies the localization of their minimum and allows to confirm the general analytical results.
- As a subject of further considerations a deeper theoretical analysis of presented results is planned. Particularly – we plan to deal with the formulating analytical conditions describing the localization of extremes of the discussed cost function.

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